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Research

Analysis of Different Hypotheses about the Geometric Pattern of the Pantheon's Coffered Dome

Abstract. This paper analyzes various hypotheses about the geometric pattern and setting out of the coffers of the Pantheon's dome. The analysis is approached from the designer's viewpoint to reveal possible methods and procedures, as well as the hypothetical assumptions and design intentions. The main hypothesis is that the designer of the coffered vault sought a geometric pattern of the coffering so that it could be perceived by the human observer as having the shape most similar to a regular squared grid with each coffer appearing to be a perfect square. The analysis also extends to the hypothetical tracing corrections to improve the perspective perception of ribs and coffers.

Introduction

This article aims to discover the set of decisions and problems the Pantheon's designers had to face when conceiving and executing the layout of the dome and coffers, both in their architectural aspects as well as regarding more technical matters of layout and setting out. The first step in discussing any hypothesis starts with the process of creating a first 3D model of the dome with our contemporary drawing means and techniques. Firstly, one has to decide which specific measurements of the dome and coffers will be taken as true in order to make this first model. The question might seem trivial at first glance, if the aim is merely to make a model that approximates the real object. If instead the work is to include a more precise analysis in order to determine the original layout hypothesis that could have been used at the moment of its construction, the problem is much more difficult or might even have no solution. Is it possible to speculate or pose plausible hypotheses if the built object does not respond to the regularity that can be attributed to it in the design? Both the Pantheon's measurements carried out throughout history with manual instruments and means, as well as more modern ones made with electronic measurement instruments, reflect the irregularity of the geometry carried out and preserved. Even addressing statistical values to obtain average measurements of the different parts of the dome we would face the layout deformations caused by structural and constructive movements that occurred throughout the monument's life. Despite all this, formulating hypotheses and checking their plausibility is interesting insofar as it brings us closer to the world and the culture in which its creators were immersed.

Problems to be solved during the design

The analysis of the geometric layout of the Pantheon's coffering poses certain questions as to the building design[1] related to the decision-making about certain aspects, such as:

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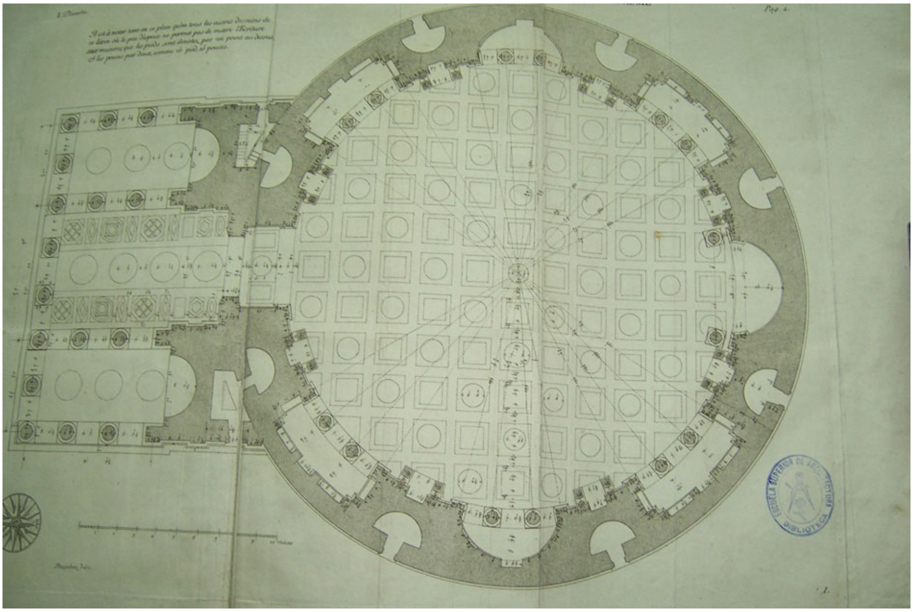

Fig. 1. Pantheon plan drawing by Desgodetz [1779], with many dimensions for pillar springers at ground level

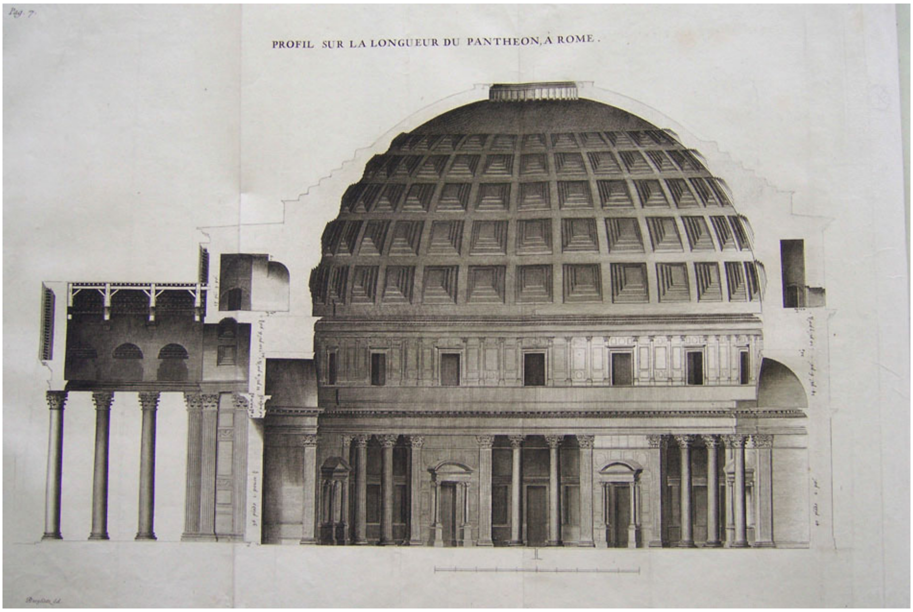

Fig. 2. Pantheon cross-section drawing by Desgodetz [1779], without dimensions

- Number of segments or meridian divisions;
- Number of coffer rows;
- Oculus diameter dimension;
- Length of the stretch without coffers;
- Proportion between the meridian rib width and the coffers width;
- Proportion between height and width of the starting coffer and consecutive coffers;
- Dimensional variations in the parallel ribs;
- Meridian rib convergence;
- Vertical and horizontal slopes of coffers;
- Measurements of the coffer background squares;
- Position of the sphere's center in relation to the springing cornice.

These are questions that every designer has to face when trying to draw this coffered dome, just as any architect or engineer usually does today when a 3D model is used to develop the designs. The dome designers must have considered these very same matters in their time (figs. 1 and 2), which lead us to ask ourselves a series of questions related to two fundamental aspects:

- Which geometric methods were used during its conception and design?
- Which construction and setting out methods were used during its construction?
- What was the geometric and arithmetic knowledge at the time and which parts could be used in the layout?
- Which were the dimensional tolerances relative to metrics and metrology?

All these questions are related to objective matters of the layout. However, from a detailed analysis of the real layout of the Pantheon, intentions related to the visual perception and the corrections carried out in the layout to improve or direct human perception of architecture in a certain sense could have existed. Greeks had already done this in the Parthenon and it was a known subject and concern to take into account in such a notable building. We should therefore ask ourselves further questions such as:

- Which layout corrections were carried out in order to improve the visual perception of the coffering?
- Was a method of perspective or projective geometry used in the layout?

The present work studies these matters and considers whether an accurate answer to it can be reached, if we are limited to stating a hypothesis that is more or less plausible can be stated, or if only mere speculations can be drawn.

Metrology and surveys

When it comes to solving all the questions previously specified in the introduction and in relation to the layout decision-making, we are forced to establish certain hypotheses about the real magnitudes of the Pantheon measurements so that they can later be checked against the actually built layout. An obstacle in our scientific path towards the reliability of any hypothesis that we may consider is found here. Even starting off with an absolute certainty, we will have to resign ourselves to a plausible hypothesis, and to assessing the probability or plausibility of the different assumptions. Metrology matters affect not only the measuring instruments and their calibration, but also the execution methods, the size and shape of things, which implies uncertainties in the setting out, and the measurements actually implemented. In the buildings built today we can observe a difference of measurements between the drawing plans and the works actually built. This difference can be greater or smaller depending on the materials, the constructive techniques applied and the measurement devices used, without getting into assessing the skill of the designers and workmen. All this, results in a series of dimensional tolerances considered in the design, which serve as contractual document when accepting or rejecting the measurements of the works actually carried out and their deviation from the project. This problem takes place in today's construction (although limited to certain tolerances), and it happened to a greater extent, in ancient constructions where concerns about dimensional tolerances were minor, since in situ construction was more usual than the prefabricated construction. Three types of mistakes or tolerances regarding measurements made by the builders of the Pantheon can be distinguished, depending on the different work phases. Without being too thorough, some of these quite common errors will be studied here:

Before the work:

- Measurement errors derived from the layout on parchment with the drawing tools from that period: wooden rulers, compasses, set squares, pencils, etc.;
- Errors derived from the measurement units used and their fractions: Roman foot, ½ foot, ¼ foot, etc., since no decimal numbers were used in the rounding off;
- Errors derived from the unfolding system of double curvature surfaces (sphere).

During the work:

- Errors derived from the setting out, such as accumulation of partial errors in the addition of measurement units such as yards or feet;
- Errors derived from the measuring equipment, such as the dimensional variation of the setting out ropes by deflection and elongation, especially in great lengths such as the radius and diameter of the dome;
- Errors in the leveling tables;
- Errors in the setting out when placing formworks.

After the work:

- Rheological movements in curing and setting;
- Settlement derived from the slow setting of hydraulic limes;
- Structural strain derived from quick load transfer and elastic deferred stress;
- Structural strain derived from irregularities in the construction and the materials strength;
- Settlement of the dome drum;
- Possible settlements at the foundation;
- Earthquakes.

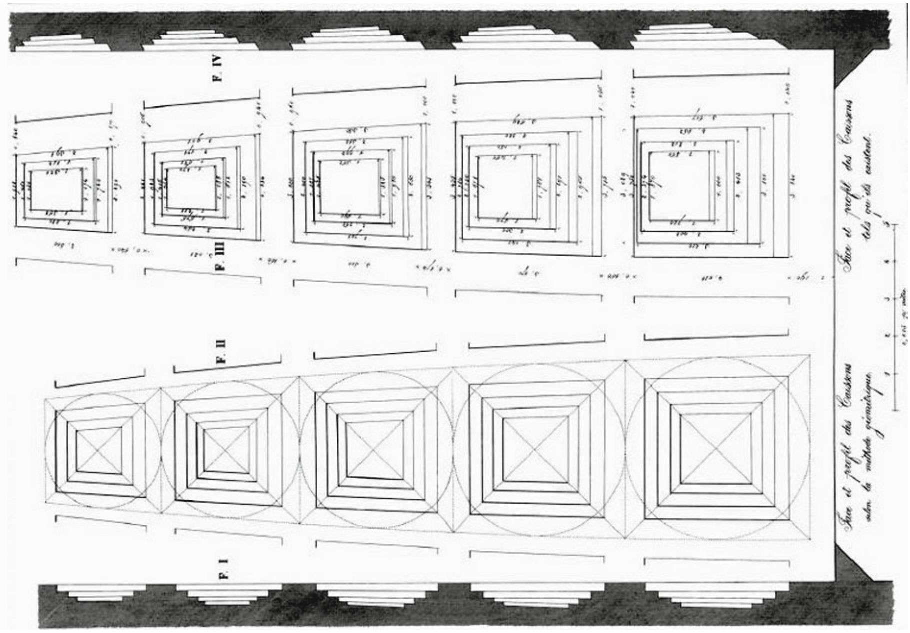

Fig. 3. Drawing and exhaustive dimensioning of the coffers performed by Rondelet [1860]

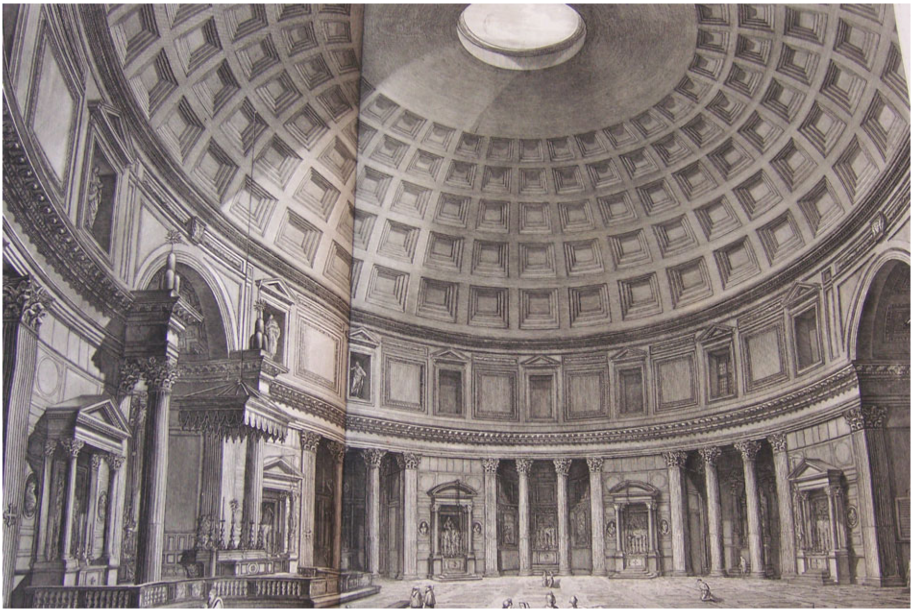

Fig. 4. Interior view of the Pantheon drawn by Piranesi [1836]

Some of these movements and strains of a structural origin were due to the masonry supporting loads once the shoring, posts and formwork had been removed have already been dealt with by Vogel [2009]; at the same time, modifications that occurred as a consequence of historical interventions in the building should also be added, such as the level of the existing paving which is raised 1.15 m over Hadrian's original paving [MacDonald 2002]. As a result of all this, the probability that the Romans, with their means and constructive techniques, had built a quasi-perfect hemisphere of this size or with a high level of precision and adjustment of tolerances, is practically non-existent. Not only this, but most probably the number of errors in quantity and disparity throughout the dome will move within a quite broad range of values.

This explains the great disparity in the measurements provided by the different authors who have carried out some kind of measurement or survey – more or less thorough – of the dome throughout history. From the first estimations by Serlio [1551] and Palladio [1570], which helped them to draw the first plans, elevations and cross-sections we now know, to the subsequent and more precise surveys by Desgodetz [1779], Piranesi [1836] and the increase of information on the coffered vault supplied by Rondelet [1860] in the nineteenth century (fig. 3). The contribution of new measurements continued in the twentieth century, becoming public with de Fine Licht [1968], MacDonald [1976], Geertman [1980], Saalman [1988], Pelleti [1989], Martínez [1989], and finally in the twenty-first century we have new data sources provided by Zaccara [2007], Sherer [2009], Graßhoff et al. [2009] and the most recent photogrammetry survey by Liciana and Altozano [2011]. However, the most reliable sources would be Graßhoff et al. [2009] and Sherer [2009], both of whom are collaborators on the project known the Bern Digital Pantheon Project [Digital Pantheon], insofar as it has been a tachymetric surveying. In addition, each surveying entails inherent errors derived from the measurement system and tools. Even more precise procedures face problems of profiling and definition of edges, both for the photogrammetry and tachymetric surveying. At the same time, the curvature problems which present differences in measurement between the arc and the chord when giving values of the coffers cannot be avoided.

A seemingly easy measurement such as the diameter in the dome springer rows presents the same construction irregularities problem. The values given by different authors vary from 43.80 m (148 Roman feet) to 44.08 m (see table 1 below), although they do not explain how many measurements in different positions of the diameter have been made, nor which type of average value or statistical value has been done of the sampling. We are clearly faced with a case of data disparity that requires a thorough data collection and a statistical treatment of these data in order to give average values of the theoretical diameter of the original project.

Perhaps the most illustrative graphic vision of the measurements and irregularities problem that we face in order to rebuild a theoretical model of the dome is the one provided by Sherer [2009]. In his virtual reconstruction of the dome the high number of irregularities or deviations of the theoretical sphericity is clearly visible, as well as the appearance of dents in big areas, which needs to be considered. This explains why certain authors locate the center of the hemisphere under the cornice line: – 0.91 m [Bartoli 1994]; –0.318 m [Pelleti 1989; Martines 1989]; –0.71 m [Aliberti and Altozano 2011].

In order to attempt a virtual reconstruction of the theoretical model of the original project of the dome, statistical values relating to at least the following global measurements would be needed:

– Curvature and radius of the sphere;
– Maximum diameter in equator;
– Position of the sphere center;
– Position of ground level.

In order to proceed with the virtual reconstruction of the theoretical model of the coffered vault, we would also need at least the statistical data of the following local measurements:

- The latitude of coffers startups;
- The coffers edges in the sphere intrados;
- Coffer meridian sloping;
- Coffer parallel sloping;
- Coffer background square;
- Parallel ribs;
- Meridian ribs, convergence and variations.

In spite of all these difficulties, and because of the impossibility of relying on non-distorted values by the stresses and general movements of the structure, there is the possibility of formulating layout hypotheses based on the analysis of rates between different measurement sources, and on this basis, confirm which one of the layout hypotheses would most likely approximate the methods used by the Pantheon designers.

Division into 28 segments

One fact that has drawn the attention of different architects, historians and researchers who have worked on the layout of the coffered dome of the Pantheon, is the apparent contradiction between the symmetry of the barrel base of the dome, and the dome itself. To put it another way, why is there a division into 28 sectors (4x7) instead of a division into 32 (4x8) or 24 (4x6), in accordance with the symmetry of the underlying drum?

Michelangelo Buonarroti, architect of the dome of St. Peter in Rome [Heene 2004] said the rotunda and the dome of the Pantheon were the work of different architects.[2] Indeed, something more specific than just an impression is the fact that the three brick arches incorporated as reinforcing in the wall of the drum at the start of the dome had been cut by carving, since their positions interfered with the formwork constructed in concrete caissons [Knell 2009]; this clearly demonstrates that this phase of the wall of the drum was constructed in the same format and symmetry as the bottom, without considering that their positions would interfere with the construction of the coffers. This would not have happened if the architect who designed the dome and the drum had been the same person; the inner face of this relieving arcade would simply have been removed. Even the design of this group of relieving arches is the same in the lower body portion than in the dome portion that hides the coffers.

For some authors [MacDonald 1976] the modification or reconstruction of the Pantheon would have been initiated by the emperor Hadrian as architect – since he had intervened in earlier works such as the Temple of Venus and Rome – and the works continued after the death of Trajan by the architect who accompanied emperor Trajan in all his wars: Apollodorus. This hypothesis seems most likely, although this historical discussion is not relevant in this analysis.

According to Heilmeyer [1975] and Lucchini [1996], Apollodorus of Damascus would have been commissioned to follow Hadrian's work after the fire of 110 A.D., which had destroyed the Campo Marzio. Trajan died in 111 A.D. and Hadrian became his successor in July 118. Hadrian was not only an architect, but also knew astronomy and astrology. Perhaps this last Emperor's esoteric choice may have influenced Apollodorus's decision to use the number 7 in the division of the vault. Why use a division into 28 areas breaking the Roman tradition, in this sense? Common Roman tradition in plotting radially symmetrical buildings derived from the simple division into two, with the most commonly used ones being the 4 (an inscribed square) and 8 (a double square rotated 45°, ad quadratum). Also according to this Roman tradition ratios commonly used in architectural composition were of 1:2, 2:3, but for divisions, computations or modulations with amounts involving a massive quantity of elements the Romans formed quantities in groups of 5 and 10. The division by 7 is somewhat unusual in the Roman world, but it was quite familiar in the Pythagorean milieu, and was not without a certain esoteric character. The cylindrical drum level follows these proportions and Roman symmetries; even the oculus measurement (1/5 of the dome diameter) fits in the Roman world. The five coffer courses also fit into this series. Only the partition into twenty-eight segments of the equator is out of the norm and uses number 7, while placing it agreement with the 8: a curious combination.

There are studies [Hannah and Magli 2009] that analyze the dome geometry and its coffers from an astronomical point of view, trying to justify the design of the dome as a kind of astronomical sundial without reaching conclusive arguments, except for those derived simply from its astronomical alignment north-south (although its actual axis deviates by about 5° in the direction of true north westward). Interestingly, from an astronomical point of view, the division in 28 meridians has more to do with the lunar cycles than with the movements of the sun. The moon has cycles that are quite close to twenty-eight days (equivalent to four weeks of seven days), with their respective positions of the full moon, waning, crescent and new moon. Moreover, the originally intended function of the building – although it was ultimately used as a tribunal – was as a temple to the Roman pantheon of gods associated with the cult of the dead, a world that has historically been associated with the moon.[3]

Beyond this discussion of who was responsible or why they decided to make a division into twenty-eight segments, what really interests us is the geometric solution to make possible or to relate the division by eight of the support drum and the division into seven parts of the starting base circumference of the dome, to match their respective symmetries, as well as Euclidean geometric design problems. This is what will be discussed below.

The most recent studies in this direction, starting from the assumption made by Heene [2004], based on the division into twenty-eight sections establishes the positioning of the coffer axes on the chapels at 45° and the central access axis, aligning the axis of one row of coffers with the access axis, and at the same time, two rows of coffers situated along the width of chapels, placed at 45° from the main axis. If the division had been 32, an additional column of coffers per quarter-sphere would have positioned the sides of the chapels at 45° between axes of coffers columns, maintaining the center axes of the coffers on line with that of the the pronaos access.

The research of this paper does not extend to numerical meanings – cabalistic or Pythagorean [Lucchini 1996] – that the division in 7,[4] or the use of number 7 might have assumed in Hadrian's times, where they had already been used in some cases, though few. Here we are only interested in the purely geometric relations of symmetry and proportion this partition implies.

The symmetry of the Rotonda is rooted in the classic square rotated 45° (ad quadratum), widely used in the Roman world at the time, both in architecture and decorative geometry. The vertices of one of these square generators coincide with the axes of the circular section chapels on the ground floor, while the vertices of the other rotated square coincide with the quasi-rectangular section chapels and the main access door.

Using the Euclidean geometry of compass and straightedge, a straight line is drawn, and over it, a perpendicular line is drawn; a circle is drawn and a division into four equal parts and a regular square have been made: on each of its sides a perpendicular line is traced at their midpoints and thus a circle divided into eight equal parts is achieved. Continuing with this simple system of division into two parts of a segment, from the initial square we can derive divisions of regular polygons or circles numbering 8, 16, 32, 64,… parts, which can be summarized in the expression ($4n$), where n is a natural number; this is equivalent to rotating the initial square a number of times maintaining the regularity of the gaps between its vertices. To achieve a division into twenty-eight parts, we should find the regular polygon with fewest sides originating, just as described, the generation of polygons derived from the square.

In the case of a division numbering 28, the regular matrix polygon is an heptagon, or seven-sided polygon which completes the twenty-eight divisions by generating four turns (7x4). The problem arises because the division of the circle by 7 is not a Euclidian partition (nor are divisions by 9, 11, 13, 19, etc.) in the sense that it is not possible to construct it with accuracy using straightedge and compass as plotting instruments, which were common drawing procedures among Roman architects and engineers, as recounted by Vitruvius [Jacobson 1986]. This problem had already been considered by Euclid when dealing with regular polygons and was subsequently addressed by Archimedes when trying to resolve some of the classical problems of Alexandrian geometry, such as squaring the circle, doubling the cube and trisecting the angle. Archimedes used a spiral for resolving the division of a right angle into an arbitrary number of equal angles (in our case 7). The problem is, as in the angle trisection solution proposed by Hippias of Elis (fifth century B.C.), that these spirals are impossible to contruct precisely with a straightedge and a compass; they belong to the field of continua in mathematics. Archimedes developed a procedure for solving the area of a parabola through recurring geometric constructions; today such problems are solvable by calculation of limits and differential calculus, but that at that time, there was no exact solution. (Many historians of mathematics consider Archimedes's methods as presaging differential calculus long before Newton and Leibniz.)

However, it should be borne in mind that in the field of construction, mathematical or geometric accuracy is neither a need nor an obsession. What can really be worrying is the degree of error that we are committing based on the metric tolerances assumed, depending on the type and scale of the work, and referring to the allowable limits of relative and absolute tolerances. Thus, the problem of the division by 7 would be perfectly acceptable if the errors are assumable during the construction work. We do not know exactly what the method of division in seven parts was used in this building, but we do know that at that time, several approximate methods for drawing a regular polygon with seven sides with more than enough precision for a masonry construction were known. An idea of the geometric knowledge of this time can be found through the translations of the work of Archimedes into Arabic by authors such as Thabit ibn Qurra (836-901 A.D.), who recovered much, and the Alexandrian geometrician Menelaus (ca. 70-140 A.D.), who is especially interesting because he lived during the construction time of the Pantheon, and thus his knowledge could have been available to the Romans of the period.

To get an idea of the error made in one of these approximate layouts of the heptagons as illustrated by Martines [1989], in which the side of the heptagon is obtained from exact layout of the regular hexagon (fig. 5), two values for a circle of unit radius are shown:

- The side of the heptagon as numerical construction supplied by a standard CAD program = 0.868 ...
- The side of the heptagon as an approximate construction from a regular hexagon = 0.866 ...

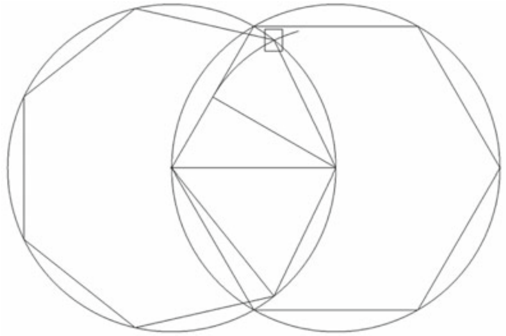

Fig. 5. Drawing the heptagon from two regular hexagons: placing the compass on the apothem of one of the overlapping triangles of two regular hexagons, an arc is drawn to intersect the circumference that will circumscribe the supposed regular heptagon

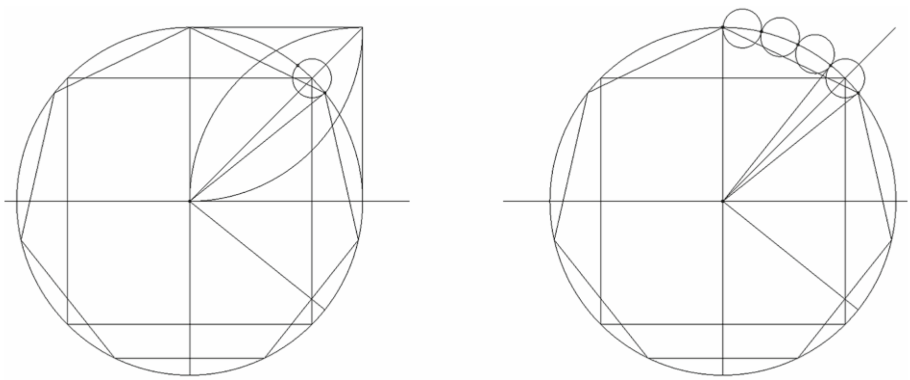

Fig. 6. A circumference is divided into four and eight parts by successive subdivisions into two. On the arch, 1/8 of the circumference is divided into seven equal parts. The apex of the heptagon will be placed adding the prior division into 1/7 to other seven equal arc parts

The ratio obtained between the two quantities is: 868/866 = 1.002309 ... The error is about 0.2% in relative tolerances, equivalent to an absolute tolerance of 2 mm in 1 meter, or 2 cm by 10 m, which in the case of the Pantheon would be an error of 44 cm in the radius, recalling for example, the difference between measures provided by Luchini [1996] 43.80 m, and Heene [2004] 43.30 m, which is 0.5 m. There is no doubt this ratio provides more than enough tolerance, not only with regard to ancient buildings but still today. Consequently, whatever the approximate method used by designers to split the dome into twenty-eight segments, the margin of error is perfectly acceptable.

In any case, no matter which the method followed for the layout of the regular heptagon, the compatibility between the division into 7 and division into 8 is striking, since this division coordinates with the division by 28 finally chosen for the layout (fig. 6). First the square inscribed in the circle is drawn and then, the heptagon is drawn by the approximate procedure (fig. 7). The distance between the division into 8 marked by the inscribed square in relation to the vertical diameter intersecting point with the circumference, and division by 7, from the same intersecting point, is the value of the radius of the circle which will serve to divide the arc sector into four parts corresponding to the division by 7. Thus, the division of the circumference in 28 parts is obtained. This procedure – or perhaps a similar one – could have been used by the designer of the Rotonda for the layout of the divisions of the starting circle at the Rotonda, and to obtain the input data to solve the setting out of the geometry of the coffers, as will be seen below:

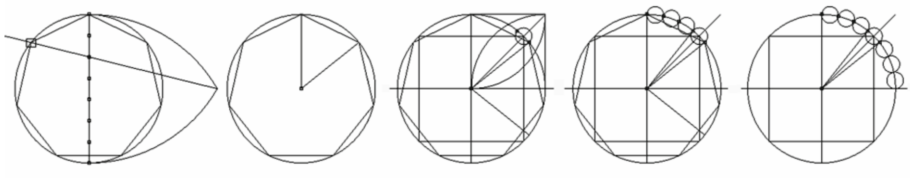

Fig. 7. The relationship between division by 7 and 8, from the approximate division by 7. The geometric construction is not exact but the relationship between 1/7 of the arc of the circumference and the relationship between the four small circles and the arc that defines the heptagon side, is accurate

About the layout and setting out of coffered vault

Once the hemisphere has been divided into 28 radial sectors and several horizontal rows of coffers, the designer must address questions such as: What is the position and the relationship between the width of the coffer starting from the first row and the width of the meridian ribs at the springer row? What height, latitude or ratio assigned to the coffers should be given? What width of the corresponding parallel rib, separating the next row of coffers should be stated? This would be the equivalent to setting the rib length and rib width,[5] although it is possible (and perhaps simpler geometrically) to make an axial layout with both coffers and ribs. In any case, the designer would try to find a method or geometric layout to facilitate, organize and coordinate all these decisions. These decisions not only affect the metric layout but also the proportions and possible perspective intentions, as well as the visual perception of the coffers. One of the first intentional decisions would be finding a maximum regularity in the layout: could the coffers really be – or be perceived – as perfect squares, or nearly so, on the intrados of the vault? The main difficulty lies in the fact that the horizontal width of the coffers will decrease and hence, the upper horizontal side will obviously be different and smaller than the lower horizontal side, which inevitably forces the designer to work with spherical trapezoids instead of squares. Even though this is inevitable, we can establish that the trapezium is quite close to a square, which leads to the condition that a circle can be inscribed inside, tangent to all sides. This problem was solved in planar geometry by Archimedes in his treatise of tangent circles [Vernet 1968] (fig. 8). The challenge the designer faces is that he must solve the problem of this layout based on tangent circles on the surface of the double curvature of the sphere (fig. 9).

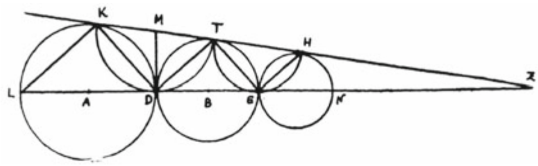

Fig. 8. Drawing used for the solution and demonstration of tangency of circles between two converging lines, from the treatise on tangent circles attributed to Archimedes [Vernet 1968]

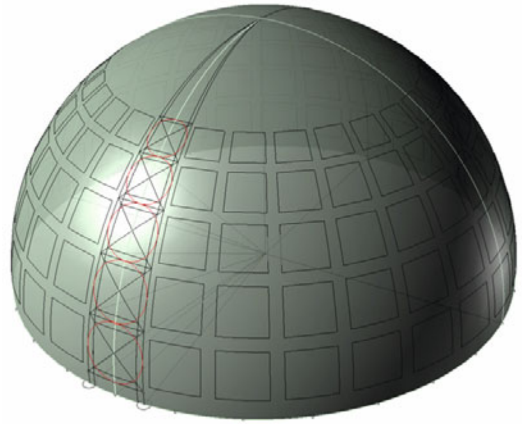

Fig. 9. The challenge the designer faces is solving the problem of the tangent circles layout on the surface of the double curvature of the sphere. There is only one correct solution using spherical trigonometry. Computer rendering: author

A simple solution would solve the problem of tangent circles on the plane of the equator to subsequently perform a perspective projection on the surface of the sphere. This procedure has been analyzed by Bartoli [1994], but checking it against the actual measurements of the Pantheon coffered vault shows that the results are far from the existing measures and the alleged regularity in the relationship of the coffer latitude-longitude, so that circles could be inserted in its intrados (fig. 10).

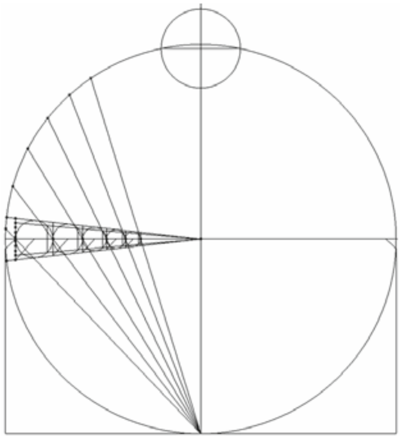

Fig. 10. Once the tangencies of circles of the equator plane are solved, a maximum circle will be projected [Bartoli 1994]. However, the deviations from the actual coffers are remarkable

Applying other methods of projective geometry may also have been possible accumulating equal length segments on the tangent to the circumference and then joining the ends of these segments with the center of the circle dividing the sphere (fig. 11).

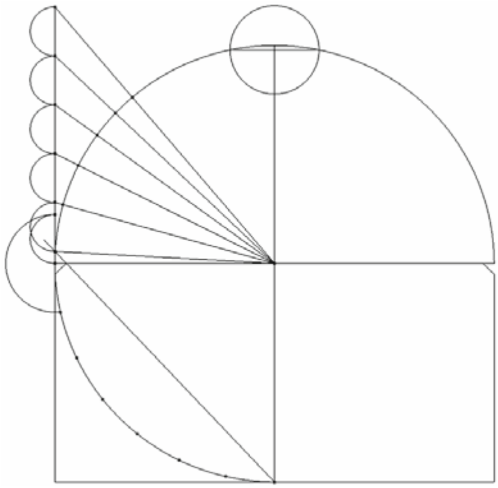

Fig. 11. Projective procedure from a regular division practiced at the vertical tangent. It is closer to the constructed shape but it remains far from approaching tolerable values

However, the simulation procedure has not produced acceptable results when they have been compared to the real measurements of the coffers.[7] Therefore, some of the measurements reported by several authors of the dome diameter at the equator (table 1) have been used, together with measurements of coffers and ribs (table 2). The sequence of reference ratios is taken from the measurements of Rondelet [1860]. According to this measurement, the ratios of the different meridian inter-axes from different coffers and rows, from the bottom row to the upper rows would be: 1.03 / 1.08 / 1.12 / 1.16 (table 2).

PANTHEON DOME METRICS	Diameter of equator	
	Roman feet	meters
MacDonald / de Fine Licht 1965-66	148	43.80
de Fine Licht 1968	148	43.80
Geertman 1980	148.5	43.92
Bartoli 1994		44.08
Geertman 1980		43.30
Pelleti / Martines 1989		44.08
Aliberti / Altozano 2011		43.80
Luchini 1996	148	43.80

Table 1. Diameter measures of the equator dome according to several authors

COFFER METRICS		Coffers starring upon cornice	Coffer base width	Coffer height	Width of parallel ribs	Meridian lower rib width	Meridian upper rib width	Meridian distance inter-axis	Meridian ratios inter-axis
Aliberti / Altozano 2011									
	Row I		4.02	4.06					
	Row II		3.90						
	Row III		3.47						
	Row IV		3.02						
	Row V		2.36						
Rondelet 1860		1.18 / 1.19							
	Row I			4.02	1.19	1.04	1.04	4.88	1.03
	Row II			3.87	0.86	1.04	1.06	4.73	1.08
	Row III			3.50	0.87	1.00	0.96	4.37	1.12
	Row IV			3.03	0.86	0.95	0.91	3.89	1.16
	Row V			2.50	0.84	0.87	0.84	3.34	
Luchini 1996		1.14 / 1.15							
	Row I			4.02	1.15				
	Row II			3.87	0.88				
	Row III			3.59	0.88				
	Row IV			3.08	0.88				
	Row V			2.50	0.83				

Table 2. Different measures for coffers according to various authors

The correct geometrical solution – closest to the measurements actually constructed – keeping this coffer regularity condition is unsolved in planar geometry of plans and sections, as it would have to unfold the plan of the spherical segments; this is geometrically and mathematically impossible due to the property of double curvature of its surface. Single-curved surfaces can be unfolded but not double-curved surfaces. Thus, the correct solution involves applying certain knowledge of spherical trigonometry. The solution is simply to use the spherical bisector between the parallel and the meridian, as had already been suggested by authors such as MacDonald [1976], and more recently Waddell [2008], although they do not explain how the Romans were able to address this type of design in the plan, before constructing the dome, which is the real problem to be solved. It should not be forgotten that the dome was built with concrete in situ, which involves the use of formwork. This implies that these forms had to be drawn and built before the construction work on the vault began, that is, there had to be a geometric procedure approaching the unfolding of spherical segments to define the measures of coffers pattern.

To ensure the geometric regularity of a hemispherical dome, only a fixed center and a radius are needed. Once constructed, it is relatively easy to draw on its intrados surface the parallel springer row and corresponding meridians to the division in the desired number of segments. On the intersecting point between any of the meridians and the equator or the maximum base circle chosen as a starter of the pattern, an arc built on the spherical surface can be traced, with a radius equal to the distance to the next intersection meridian-base circle. This arc will intersect the meridian arc on which the center of the arc lies. The midpoint of this arc is marked by other arcs centered at the end of the first reference arc and this point will serve to draw the spherical bisecting joining this point with the primitive arc center and extending to the intersection with the next meridian (fig. 12).

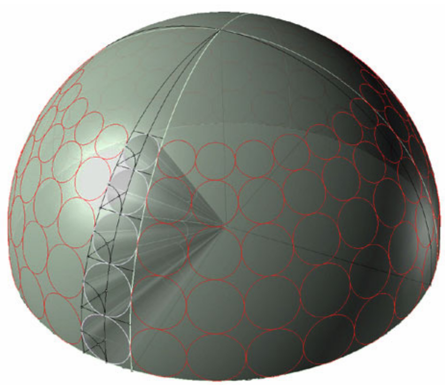

Fig. 12. Drawing of circles tangent to the sphere through the process of spherical bisector. Computer rendering: author

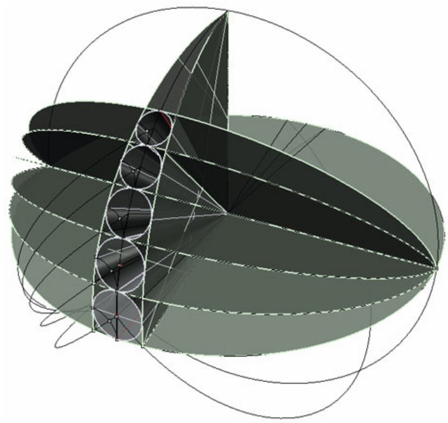

Fig. 13. The circle inscribed in the spherical trapezoid is the result of the intersection produced between a cone with its center in the sphere and the surface of it. Computer rendering: author

This intersecting point will serve to pass a maximum circle turning on the equator meridian of the dome. This will give a spherical trapezium formed by maximum circles that fit inside a circle with both respective tangent points on each of the maximum circles. This circle inscribed in the spherical trapezium is the result of the intersection between a cone produced in the center of the sphere and the spherical surface (fig. 13). This procedure can become recurring for the next row of coffers and so on. If the intersection points of the respective spherical bisectors are joined and an interpolation is performed, the result is a rhumb line (loxodrome) that will asymptotically converge towards the north pole or top center of the dome. This rhumb line can be laid out by points – as many points as needed – but it cannot be drawn using the methods of Euclidean geometry; the same thing is true of the Archimedean spiral.

However, it should be noted that the coffered vault is traced with meridians and parallels, so this geometric construction with which the bisectors were obtained must be rectified by passing two parallel lines through the points of upper and lower tangency of the spherical trapezium, replacing the maximum circles used for the layout. This way we could get a fairly accurate layout of the coffers. In fact, this would have been the old procedure used in the decoration of geometric domes, where a built spherical surface was available, on which they could be traced directly using a rope and a burin. But this is not the case of the Pantheon, because it is necessary to trace the geometry of the coffer formwork before building the dome surface.

Consequently, Pantheon designers were forced to develop some geometric method of unfolding spherical segments in order to set out and build the formworks that would shape the coffers. They could also have used a numerical procedure based on spherical trigonometry, as developed by Menelaus of Alexandria [Krause 1936] in his treatise *Sphaerica*.[8] The designers of the Pantheon could have known this marvelous treatise, but even for an architect today it is a difficult book to understand and manage. Therefore, we tend to assume that either they were assisted by some astronomer of the time (one initiated in this scientific knowledge) or, more likely, it can be assumed that they used a simplified geometrical approach to proceed with the design and layout.

About the coffer pattern based on the conical unfolding of the sphere

In this section, the hypothesis that the Romans could have used a method of unfolding the spherical surface of double curvature conical sectors by simplifying spherical segments is studied, as this allows for a correct unfolding due to its characteristic of simple surface curvature. In the documents of the time, there is no evidence of such a geometrical procedure, however, it is a possible scenario corresponding to the knowledge of Euclidean geometry of the historical period in which they drew and built the Pantheon. This consistency with Euclidean geometry is what allowed Alonso de Vandelvira (1544-1626), in his treatise on architecture [Barbé-Coquelin de Lisle 1977], to propose a graphical method to solve the stereotomy of the ashlars of a hemispherical dome. His method is based on the substitution of the spherical surface by intersections between conical surfaces between parallels [Vandelvira 1573: 61] (figs. 14 and 15). In this case, the starting point is a regular partition of meridians on the maximum circle. A cone is drawn with base on the lower meridian from the equator and incorporating the upper meridian in its surface, thereby obtaining the generatrix and height of the said cone. The other meridians are obtained in a similar way. On the equator circle, as many sectors as desired are established; in the case of the Pantheon, this would be 28. This makes it possible to numerically calculate the circle arc length corresponding to that partition. This arc length allows the development of the conical sector by simply following this procedure. Once the generatrix of the cone is obtained as explained above, it is used to draw an arc with base on the cone apex, starting from the lower meridian and with equal length to that which had been calculated numerically for the sector. These two arcs, limited by both generatrixes acting as radius of the sector, constitute the plane unfolding in the conical sector that replaces the corresponding spherical sector. This unfolding allows the designer to make tracings or drawings as needed, with the ease of working in true real dimension and then reintegrating this layout to the simplified surface of the dome.

This system, easy to use and understand by any mason, is very useful for building the necessary bevels to perform the stone-carving of the dome ashlars with a perfectly acceptable margin of error for the building system, when trying to get voussoir ashlars to build domes. However, as the number of horizontal courses or meridian partition decreases, error increases until it exceeds acceptable limits for use in construction.

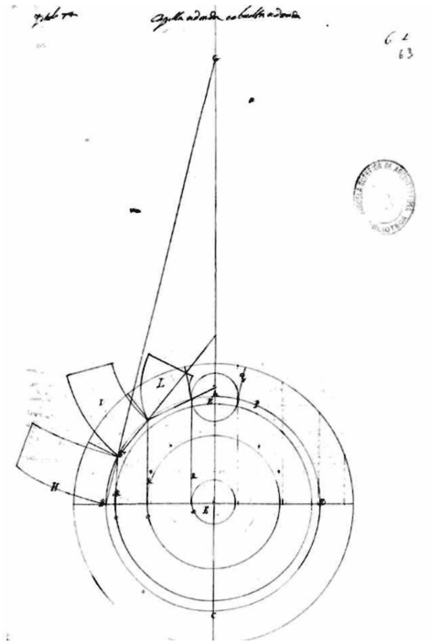

Fig. 14. Vandelvira's tracing procedure, projecting cones on the sphere as it ascends by the coffer parallels. The section of the cone is unfolded on the plan

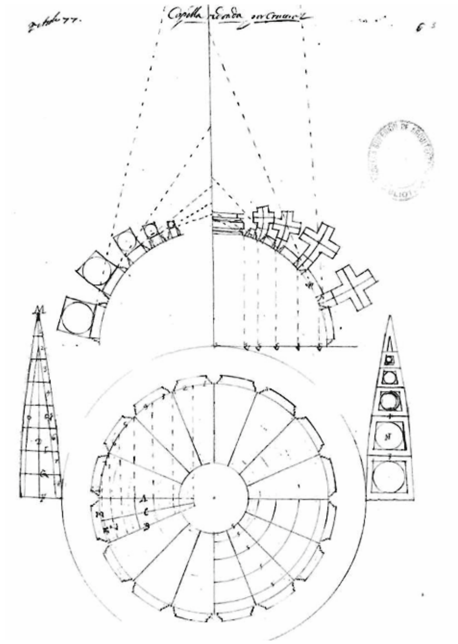

Fig. 15. Vandelvira's tracing procedure for obtaining the stereotomy of the ribs in a coffered dome

Another issue to consider – one which is a key concern in the layout of the Pantheon coffers – is the uncertainty produced when drawing the first cone on any meridian or the equator, in compliance with the condition of containing a circle in the unfolding of its conical sector (as we proposed when we performed the exact layout) based on the spherical bisectors sector where coffers are contained. The indeterminacy is based on whether the equal partition of any meridian or the equator is freely chosen by the designer, the height or latitude where the next meridian is, fulfilling the condition to inscribe a circle tangent to the unfolding conic sector boundary lines, cannot be solved graphically or by Euclidean methods. The solution is analytical and to reach it, it is necessary to solve a system of two equations with two unknowns,[9] which was not feasible until the time of Descartes.

Visual perception and its corrections

"Caesar's wife must not only be honest but must appear to be so". Romans pushed the limits in everything they did. In the Pantheon, they pushed the limits not only in terms of size but of regularity. A maximum regularity was sought in the Rotonda: the coffers were to be as regular as possible, and were also to appear so. Allowing for some differences in the height of the observer, whose ideal position was in the centre of the floor, each and every point of the intrados of the dome would appear to be at the same distance from him: the maximum regularity in this case was provided by the spherical shape. However, when coffered vault is observed as the observer moves through the space, the perception will not have serious corrections or obstructions preventing the full appreciation of the coffered vault, even the inclination of the coffers. Obviously, there will come a moment when the observer is so close to the interior walls that the cornice at at the springline of the dome will partially block the first row of coffers. In spite of this, a compromise is to be found, turning the inside coffer inclination to a point not far from the interior walls [Valenti 2009]. Regarding the inclination of the architrave of the upper coffers, there is more variation among individual authors, but it appears to be placed between the center of the half sphere and the vertical axis of the hemisphere. In any case it seems to respect criteria of perspective and lighting, because if the upper slope is too closed, there will be too much shadow on the bottom of the coffers. The inclination in the direction of the parallel is more regular across multiple rows, but in the fifth and last row it has to miss a step, with the prospective criterion that all of the recessed square coffers are as regular – that is, most nearly square – as possible. In fact, it is noteworthy that in the last two rows of coffers, the recessed squares are indeed nearly equal. Then, as we move down into the third, second and first rows, the bottom square increases in size but not at the same rate as the coffer contour does against the intrados of the dome surface. It seems the designer's intention was for the viewer to perceive the whole coffered vault as a regular flat grid using this small perspective trick: slightly modifying the size and format of the recessed square coffers to achieve this perspective effect. So, with a clever manipulation of the format and dimensions of the architraves of the coffers inclination, correcting the trapezoidal layout format of the coffer intrados is achieved, making them look square, although this is geometrically impossible.

The search for regularity can also be seen in the pavement, which features the regular checkerboard loved by the Romans. This pavement grid suggests a parallelism with the grid of the coffered vault; suggestion that is reinforced by the sizing of the ribs and squares: the width of the ribs of the pavement is similar to the width of the parallels and meridian ribs that surround the first coffer row.

But while in the ground plan the square grid of the ribs can maintain its regularity throughout its extension on the plan, this does not happen with the grid of the vault, which extends all over the spherical surface of the double curvature, compelling it to converge from the equator towards the pole. On a surface like this, the meridian ribs should converge toward the pole, gradually decreasing in size along the geodesic of the meridians. Similarly, parallel ribs should decrease proportionally in width, as the coffers are higher up. This should have been so, but it was not built that way; measurements of the coffered vault confirm that a geodesic sizing criterion has not been followed for the ribs, and therefore not for the coffers either. Meridian ribs converge gradually but without following the convergence of the meridians (1.04/1.04/1.00/0.95/0.87 [Rondelet 1860]). The parallel ribs also converge in the first and the second row but remain virtually unchanged until the fifth row (1.19/0.86/0.87/0.86/0.84 [Rondelet 1860]). Regarding the coffers, the dimensions decrease in proportion to the rib values, according to the rib/coffer side relationship, 1/3.5, 1/2.5 in relation to the fifth row. The coffer depth decreases, as does the thickness of the dome, from 22.8 cm to 17.8 cm [De Fine Licht 1968], which also facilitates the regular positioning of the background square without visibly affecting the width of the steps of the sloping. Piranesi drew it like that and Leclère documented it.

Clearly, the width of the ribs has been set out according to criteria far from the geodesic and trigonometrical orthodoxy provided by Menelaus of Alexandria in his *Sphaerica*. Although the designer of the coffered vault did not know Menelaus, there is no doubt that he probably knew that the width of the vertical ribs should follow the meridian segments directed toward the pole, and that the width of the parallel ribs should clearly be decreasing. If they are not, the most likely scenario is to assume that corrections were made based on optical, perspective or theatrical criteria, with the intention being to convey the maximum section of checkerboard regularly adapted to the spherical surface. It is a shame that no documents have come down to us concerning the two quarter-sphere coffered exedrae, probably designed by Apollodorus, in Trajan's Ulpia Basilica, both equal and symmetrical, 150 feet in diameter, similar to the diameter of the Pantheon, but constructed in wood.

Before the Romans, the Greeks had already experimented with these visual corrections of architectural patterns, as in the Parthenon. Romans, as heirs to Greek culture, were certainly aware of these aesthetic subtleties, so presumably they would echo them in a building as important as the Pantheon. In support of this hypothesis, Theocharis [2009] points out the possibility that the floor and porch access could have a perspective entasis. The non-specialist viewer does not have to know about geometry. What is relevant is the observer's visual perception of the building. In this case, appearance is more important than being: 'It is more important that Caesar's wife appear honest than actually be so.'

Acknowledgment

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Notes

1. The present paper was a response to the doubts posed by my students in the School of Architecture of the Polytechnic University of Madrid (UPM), when I proposed as an exercise to draw the coffered pattern of the Pantheon dome using a CAD 3D modeling standard. What at first appeared trivial, eventually turned into a genuine work of historical and mathematical research, of which some parts are presented here.
2. This is an affirmation that, as an architect, I agree with entirely because that is the aesthetic feeling I personally have (along with most of the colleagues with whom I have discussed this), regardless of my historical knowledge on the subject.
3. We might say that the dead do not look at the sun but the moon; only the moon comes out at night.
4. The division into 7 provides a numerical curiosity for both $1/7 = 0.\overline{142857}$ and its half $1/3.5 = 0.\overline{285714}$; note that an inversion of the first sequence (142857) is repeated periodically in its half (285714). Also the period – 142857 – comprises the numbers 14, 28, 57, in which 14=7x2, 28=2x14 (=7x4), and 57 is almost equal to 28x2 (= 7x8), so that each pair of numbers doubles the previous one and they are each multiples of 7. In cases like this, esoterism is fed by mathematics.
5. Instead of setting out axes, measurements are marked with the width of the coffers and then the width of the ribs, alternately. It is a normal way used by woodwork framers for setting out rafters on pitched timber roofs.
6. Applying this pattern model according Bartoli [1994], and taking a radius value of 43.80 m in the equator and a gap of 0.17 m at the start of the first coffer, we would obtain the following corresponding latitudes in ascending rows of coffers: I (4.84), II (4.56), III (4.11), IV (3.58) and V (3.03) m.
7. Applying this pattern model we would obtain the following corresponding latitudes in ascending coffer row: I (4.75), II (4.24), III (3.56), IV (2.88) and V (2.31) m., still further from what is actually built than the previous model.
8. The original Greek treatise on trigonometry, *Sphaerica* by Menelaus of Alexandria, has not been preserved but several translations into Arabic, Hebrew and Latin have survived. A list of these translations is collected by Max Krause [1936] and more recently by Knobloch [2009].
9. I am developing the correct geometrical and mathematical solution for a future article.

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